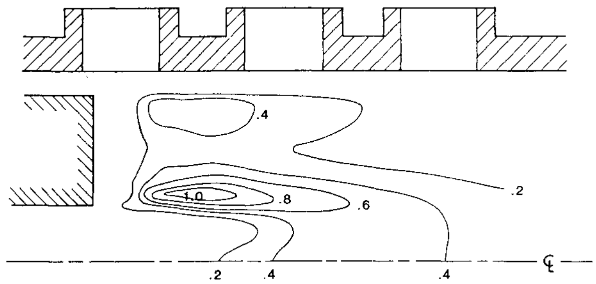

Fig. 4 Normalized eddy dissipation rate for geometry A, ideal.

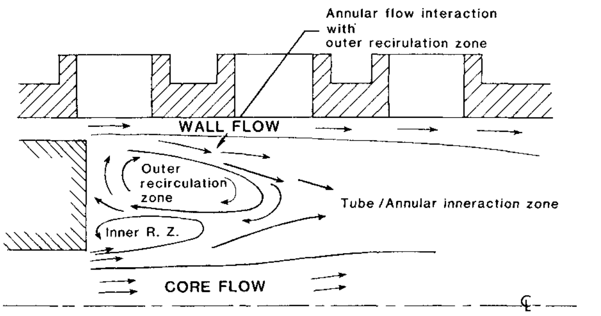

Fig. 5 Interpretation of nonreacting flowfield generated by CORA2.

generated. (Geometry R represents a combustor where a strong fuel flow or fuel/air flow is introduced into the combustor along the centerline. The inlet mass flow percent through the tube is 25 and because of the small tube diameter, the tube inlet velocity is large in relation to the annular inlet velocity.) In such instances, flow from the annular region passes the outer recirculation zone and flows into the inner recirculation zone (Fig. 3g). Geometry R, with the extreme nonideal inlet conditions, indicates that significant alteration to the fundamental flowfield can result from shifting inlet velocities far from ideal inlet conditions; nevertheless, the characteristic length $(D-d)$ still appears to represent the circulation zone length. Changes in the flowfield structure with respect to total inlet mass flow rate (fractional flow split between tube and annular flow remaining constant), temperature and pressure were examined by altering the values of these constants and generating velocity values for geometry A shrouded using CORA2. As expected, the flowfield structure remained virtually unchanged for variations of inlet mass flow rate ($0.3 < m < 1.5$ kg/s), temperature ($287 < T < 800$ K) and pressure (404-808 kPa) verify little or no variation in air properties within this range.

Many of the observations about the invariant behavior of the flowfield structure, found numerically, have been confirmed experimentally by the authors.

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Finite-Element Method for a Uniformly Loaded Cantilever Beam with General Cross Section

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Introduction

IN the analysis of beam type structures, the Euler-Bernoulli theory of bending is widely used. In this theory, the transverse shear strain is zero, and the transverse shear stress is nonzero. When the beam is short, the effect of transverse shear cannot be ignored and should be included. Numerous papers have been published in this area for beams with thin-rectangular or simple cross sections such as circle or ellipse.[1,2] If the beam cross section is not simple, and a detailed stress distribution of the entire beam is desired, a three-dimensional finite-element analysis is required, even for a cantilever beam subjected to a uniform surface or gravity load.

The theory developed by Michell[3,4] indicates that the problem of a uniformly loaded beam can be reduced to a plane strain problem if the warping displacement of a cantilever beam due to a unit transverse shear load at the free end is known. Recently, a method[5] using a two-dimensional finite element to calculate shear stresses and warping displacement of a cantilever beam subjected to a unit shear load at the free end was presented. Therefore, Michell's theory in conjunction with this method is used in this paper to obtain a finite-element solution for a uniformly loaded cantilever beam with general cross section. The warping displacement and shear stresses of a cantilever beam due to a unit shear load are first obtained using a two-dimensional finite-element model for the beam cross section. A plane-strain problem is then established with internal body and boundary forces computed from the warping displacement. This plane-strain problem is solved by using the regular two-dimensional finite-element program on the same model used for warping displacement calculation. Thus, a complete solution for the problem of a uniformly loaded cantilever beam is obtained. Numerical examples are given for cantilever beams with circular and thin-rectangular cross sections. The circular beam is loaded by its own weight, whereas the thin-rectangular beam is loaded by a uniform surface load on the upper face of the beam. Finally, the stress results from the present method are compared with the closed form solution for a circular beam and the two-dimensional elasticity solution for a rectangular beam.

Reduction of Uniformly Loaded Cantilever Beam to a Plane-Strain Problem

In this section, Michell's theory to reduce the problem of a uniformly loaded cantilever beam is briefly described. First, let the z axis be the beam axis, and x and y axes be the principal axes through the centroid of the beam cross section.

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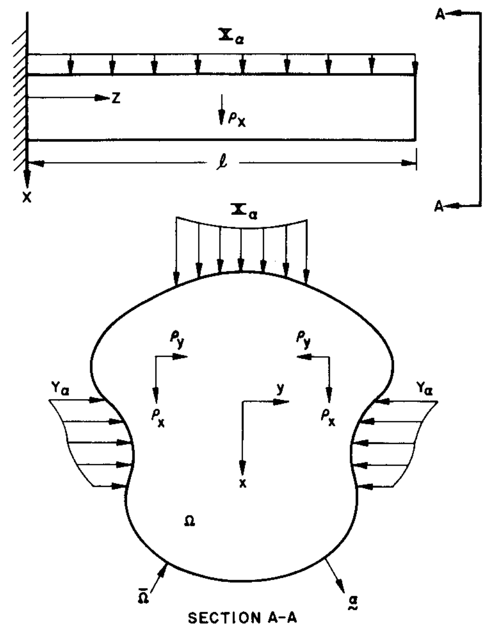

Fig. 1 A typical cantilever beam.

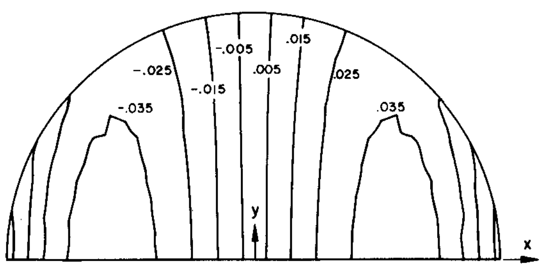

Fig. 2 Contour of σ_{xx} in a circle from present method (in psi).

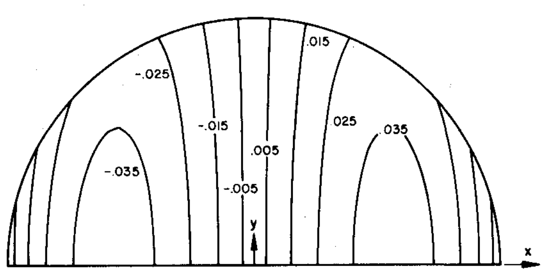

Fig. 3 Contour of σ_{xx} in a circle from closed form solution (in psi).

The material of the beam is homogeneous and isotropic with E, ν, and G, denoting the modulus of elasticity, Poisson's ratio, and shear modulus, respectively. The loadings are assumed to be uniformly distributed along the z axis and symmetric about the x axis. The loadings include the internal body forces ρ_x and ρ_y, in body Ω and surface tractions X_α and Y_α on the cylindrical surface $\bar{\Omega}$ in the x and y directions, as shown in Fig. 1. The beam is clamped at $z=0$ and free at $z=\ell$. As shown in Ref. 4, the stress and strain components of the beam for these loadings can be expressed as

$$\begin{aligned}\sigma_{xx} &= \sigma_{xx}^{(2)}z^2 + \sigma_{xx}^{(1)}z + \sigma_{xx}^{(0)}\\ &\vdots \qquad \vdots \qquad \vdots\\ e_{xx} &= e_{xx}^{(2)}z^2 + e_{xx}^{(1)}z + e_{xx}^{(0)}\\ &\vdots \qquad \vdots \qquad \vdots\end{aligned} \tag{1}$$

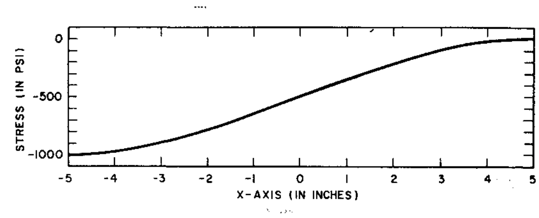

Fig. 4 Distribution of σ_{xx} in a thin-rectangle from present method.

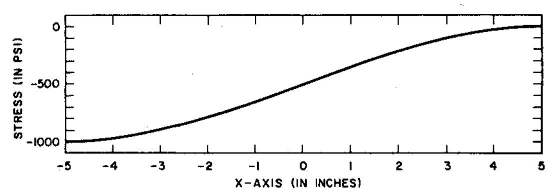

Fig. 5 Distribution of σ_{xx} in a thin-rectangle from plane stress solution.

where the coefficients of the z^2, z, and z^0 terms are functions of x and y only. By using the equations of equilibrium, the equations of compatibility, the boundary conditions, and collecting the z^2, z and z^0 terms separately, the unknown coefficients in Eq. (1) can be obtained (see Ref. 4 for step-by-step derivation). The shear-strain components for the z^2 term and $e_{xy}^{(1)}$ for the z term are zero. The normal-strain components for the z^2 and z terms are proportional to the uniform load per unit length in the direction of the x axis. The shear-strain components on the beam cross section (z face) for the z and z^0 terms are related to a warping function χ, which satisfies the two-dimensional Laplace equation in Ω and the following boundary condition on $\bar{\Omega}$:

$$\frac{\partial\chi}{\partial\alpha} = -\left[\frac{1}{2}\nu x^2 + \left(1-\frac{1}{2}\nu\right)y^2\right]\cos(x,\alpha) - (2+\nu)xy\cos(y,\alpha) \tag{2}$$

where α is a unit outward normal vector on $\bar{\Omega}$, as shown in Fig. 1. The rest of the coefficients for the z^0 term can be determined from a plane-strain problem with the following differential equations:

$$\begin{aligned}&\frac{\partial\sigma'_{xx}}{\partial x} + \frac{\partial\sigma_{xy}^{(0)}}{\partial y} + \left[\rho_x + \sigma_{zx}^{(1)} + \frac{2G\nu}{1-2\nu}\frac{\partial e_{zz}^{(0)}}{\partial x}\right] = 0 \quad \text{in } \Omega\\ &\frac{\partial\sigma_{xy}^{(0)}}{\partial x} + \frac{\partial\sigma'_{yy}}{\partial y} + \left[\rho_y + \sigma_{yz}^{(1)} + \frac{2G\nu}{1-2\nu}\frac{\partial e_{zz}^{(0)}}{\partial y}\right] = 0\end{aligned} \tag{3}$$

and the boundary conditions

$$\begin{aligned}&\sigma'_{xx}\cos(x,\alpha) + \sigma_{xy}^{(0)}\cos(y,\alpha) = \left[X_\alpha - \frac{2G\nu}{1-2\nu}e_{zz}^{(0)}\cos(x,\alpha)\right] \text{on } \bar{\Omega}\\ &\sigma_{xy}^{(0)}\cos(x,\alpha) + \sigma'_{yy}\cos(y,\alpha) = \left[Y_\alpha - \frac{2G\nu}{1-2\nu}e_{zz}^{(0)}\cos(y,\alpha)\right]\end{aligned} \tag{4}$$

where $\sigma_{zx}^{(1)}$, $\sigma_{yz}^{(1)}$, and $e_{zz}^{(0)}$ are functions of χ and

$$\begin{aligned}\sigma'_{xx} &= \frac{2G\nu}{1-2\nu}e_{yy}^{(0)} + \frac{2G(1-\nu)}{(1-2\nu)}e_{xx}^{(0)}\\ \sigma'_{yy} &= \frac{2G\nu}{1-2\nu}e_{xx}^{(0)} + \frac{2G(1-\nu)}{(1-2\nu)}e_{yy}^{(0)}\end{aligned} \tag{5}$$

Once the function χ is known, the body and boundary forces expressed in the brackets in Eqs. (3) and (4) can be calculated. The problem of a uniformly loaded beam is then reduced to a plane-strain problem with quantities σ'_{xx}, σ'_{yy}, and $\sigma^{(0)}_{xy}$ to be determined.

Reference 5 provides a finite-element method to solve for the warping displacement and shear-stress distribution of a cantilever beam due to a unit shear load. This warping displacement is actually equal to $-(\chi + xy^2)/EI$, where I is moment of inertia about the y axis. Once the function χ is determined, the internal body and boundary forces in Eqs. (3) and (4) can be numerically calculated. The regular finite-element program can then be used to solve for this plane-strain problem, and the solution for the uniformly loaded cantilever beam is completely determined.

Numerical Examples

Two cross sections are investigated. One is a circle with a radius of 1.0 in. and the other is a thin-rectangular section with a dimension of 10.0×1.0 in. The circular beam is loaded by its own weight, and the thin-rectangular beam is loaded by a uniform surface load of 1000 lb/in.^{2} on the upper face of the beam. The loads are downward along the x axis. The material properties are as follows: $E = 30 \times 10^6$ psi, $\nu = 0.25$; and $\rho_x = 0.283$ lb/in.^{3}

An in-house computer program using an eight-node isoparametric quadrilateral element is used to solve for the warping displacement and shear stresses due to a unit shear load, as well as for the plane-strain problem. Due to the nature of these problems, only half of the cross sections are modeled. The models for the half circle and the half thin-rectangular consist of 96 and 80 elements, respectively. The stresses of the entire beams are computed. The shear stresses σ_{zx} and σ_{yz} are linear with respect to z, and the in-plane stresses σ_{xx}, σ_{yy}, and σ_{xy} are not functions of z. The stresses for the thin-rectangular beam are basically uniform along the width of the beam and are averaged in order to compare with the two-dimensional elasticity solution.

The stresses computed from the present method are then compared with the closed form solutions.[4] The stress distribution of σ_{xx} for both circular and thin-rectangular beams are presented in Figs. 2–5 for comparisons (see Ref. 6 for other stress components). By comparing the results from the present method and the closed form solutions, excellent agreement between the two results is seen.

Conclusion

From the theory and numerical results presented, it can be seen that this finite-element method can accurately calculate all the stress and strain components of a uniformly loaded cantilever beam with a general cross section. The displacement field of the beam can be obtained without difficulty. It should be noted that Michell's theory is not limited to the case of a cantilever beam. For a simply supported beam, the solution is similar to that of the cantilever beam. The main advantage of this method is that it can obtain the complete solution by using only a two-dimensional model of the beam cross section rather than a three-dimensional model of the entire beam.

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